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Note

Some upper bounds for the product of the domination number and the chromatic number of a graph

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Abstract

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Some new upper bounds for $\gamma\chi$ are proved, where γ is the domination number and χ is the chromatic number of a graph.

All graphs considered in this note are finite, undirected, and without loops or multiple edges. Our terminology is based on [2]. For a graph G, let $n, \delta, \Delta, \gamma$, and χ denote the order, the minimum degree, the maximum degree, the domination number, and the chromatic number of G, respectively. Recently Gernert [5] has obtained the following two inequalities for the product $\gamma\chi$ of the domination number and the chromatic number of a graph.

Theorem A. *If G is a connected graph with $n \geqslant 5$, then $\gamma\chi \leqslant \frac{1}{4}n^2$.*

Theorem B. *If G is a regular graph with $n \geqslant 5$, then $\gamma\chi \leqslant \frac{6}{25}n^2$.*

In this note we present some improvements of the above inequalities. The following two lemmas will be useful in our proofs.

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Lemma 1. *For a connected graph G, the following inequalities hold:*

(i) ([1]) $\chi \leqslant \Delta$ *if G is neither a complete graph nor an odd cycle;*

(ii) ([3]) $\chi \leqslant 3n/5$ *if G is a regular graph with* $\Delta \leqslant n-2$;

(iii) ([6]) $\gamma \leqslant (n+1-\delta)/2$ *if G is not isomorphic to the complement of a one-regular graph;*

(iv) ([6])

$$\gamma \leqslant 2 + \frac{(n-1-\Delta)(n-2-\delta)}{n-1} \doteq \frac{f_{n,\delta}(\Delta)}{\Delta};$$

(v) ([4])

$$\gamma \leqslant \frac{n+1}{2} - \frac{(\delta-1)\Delta}{2\delta} \doteq \frac{g_{n,\delta}(\Delta)}{\Delta}.$$

Lemma 2. *For integers n and δ with* $n > \delta+2 \geqslant 4$, *let* $f_{n,\delta}$ *and* $g_{n,\delta}$ *be the functions defined in Lemma 1. Then* $f_{n,\delta}$ *is decreasing in the interval* $(\Delta^f_{n,\delta}, +\infty)$, $g_{n,\delta}$ *is increasing in the interval* $(-\infty, \Delta^g_{n,\delta})$, *and* $g_{n,\delta}$ *has the maximum at the point* $\Delta^g_{n,\delta}$, *where*

$$\Delta^f_{n,\delta} = \frac{(n-1)(n-\delta)}{2(n-2-\delta)} \quad \text{and} \quad \Delta^g_{n,\delta} = \frac{\delta(n+1)}{2(\delta-1)}.$$

Theorem 1. *If G is a connected regular graph with* $n \geqslant 6$ *and G is different from the cycle* C_7, *then*

$$\gamma\chi \leqslant \frac{(n+1)^2}{8}. \tag{1}$$

Proof. If G is a complete graph, then $\gamma\chi = n$ and (1) holds for $n \geqslant 6$. If G is a cycle, then $\gamma\chi = \lceil n/3 \rceil (2-n+2(\lceil n/2 \rceil))$ and therefore (1) holds if $n=6$ or $n \geqslant 8$. Thus we may assume that G is neither a complete graph nor a cycle. Then $3 \leqslant \Delta \leqslant n-2$ and we consider two cases. First, if n is odd or $\Delta \neq n-2$, then combining (i) and (iii) we obtain

$$\gamma\chi \leqslant \frac{(n+1-\Delta)\Delta}{2} \leqslant \max_{\Delta} \frac{(n+1-\Delta)\Delta}{2} = \frac{(n+1)^2}{8}.$$

Second, assume that $\Delta = n-2$ with even $n \geqslant 6$. Then $\gamma = 2$ and, according to (ii), $\chi \leqslant 3n/5$ (resp., $\chi \leqslant 3$ if $n=6$). Hence,

$$\gamma\chi \leqslant \frac{6n}{5} \leqslant \frac{(n+1)^2}{8}$$

for $n \geqslant 8$, while $\gamma\chi \leqslant 6 \leqslant (6+1)^2/8$ for $n=6$. □

Theorem 2. *If G is a connected graph with* $\delta \geqslant 2$, *then*

$$\gamma\chi \leqslant \frac{\delta}{8(\delta-1)}(n+1)^2. \tag{2}$$

Proof. It is easy to verify the inequality (2) if G is a complete graph or a cycle. If G is neither a complete graph nor a cycle, then applying (i), (v), and Lemma 2 we obtain

$$\gamma\chi \leqslant g_{n,\delta}(\Delta) \leqslant g_{n,\delta}(\Delta^g_{n,\delta}) = \frac{\delta}{8(\delta-1)}(n+1)^2. \qquad \square$$

Note that Theorem B follows from Theorem 1 for connected regular graphs of order at least 6 which are different from the cycle C_7. Similarly, Theorem A follows from Theorem 2 if $\delta \geqslant 4$ or if $\delta = 3$ and $n \geqslant 7$. Finally, for $\delta = 2$ and $n \geqslant 17$, Theorem A follows from the next result.

Theorem 3. *If G is a connected graph with $\delta = 2$, then*

$$\gamma\chi \leqslant \tfrac{2}{9}(n+1)^2. \tag{3}$$

Proof. A simple verification shows that the inequality (3) holds if G is a cycle. Thus assume that G is not a cycle. Then the hypothesis $\delta = 2$ implies that G is not a complete graph and $n \geqslant 4$. There are two cases to consider.

Case 1: $\Delta \leqslant 2(n+1)/3$.

Since the function $g_{n,2}$ is increasing in $(-\infty, \Delta^g_{n,2})$ (see Lemma 2) and

$$\Delta \leqslant 2(n+1)/3 < n+1 = \Delta^g_{n,2},$$

so we have

$$g_{n,2}(\Delta) \leqslant g_{n,2}(2(n+1)/3) = 2(n+1)^2/9.$$

Consequently, it follows from (i) and (v) that

$$\gamma\chi \leqslant g_{n,2}(\Delta) \leqslant \tfrac{2}{9}(n+1)^2.$$

Case 2: $\Delta > 2(n+1)/3$.

In this case n must be not smaller than 6. First, if $n = 6$, then $\chi \leqslant 5, \Delta = 5$ and therefore $\gamma = 1$, so $\gamma\chi \leqslant 5$ and the inequality (3) holds. Second, if $n \geqslant 7$, then

$$\Delta^f_{n,2} \leqslant 2(n+1)/3 < \Delta$$

and therefore

$$f_{n,2}(\Delta) \leqslant f_{n,2}(2(n+1)/3)$$

(by Lemma 2). This together with (i) and (iv) gives

$$\gamma\chi \leqslant f_{n,2}(\Delta) \leqslant f_{n,2}(\tfrac{2}{3}(n+1)) \leqslant \tfrac{2}{9}(n+1)^2. \qquad \square$$

Concluding remarks. For integers $r \geqslant 1$ and $k \geqslant 2$, let G be a graph formed as follows: Take $n = rk + k$ vertices $x_1, x_2, \ldots, x_{rk}, a_1, \ldots, a_k$ and join x_i to x_j, $1 \leqslant i < j \leqslant rk$. Further, join a_i to the vertices $x_{(i-1)r+1}, \ldots, x_{ir}$, $1 \leqslant i \leqslant k$. In the resulting graph G we have $\delta = r, \chi = \delta k, \gamma = k$, and

$$\gamma\chi = \delta k^2 = \frac{\delta}{(\delta+1)^2} n^2. \tag{4}$$

For positive integers n and δ with $n > \delta$, let $B_{n,\delta}$ be the smallest integer B such that the inequality $\gamma\chi \leqslant B$ holds for every connected graph of order n and minimum degree δ. It follows from (4) and Theorem 2 that

$$\frac{\delta}{(\delta+1)^2} n^2 \leqslant B_{n,\delta} \leqslant \frac{\delta}{8(\delta-1)}(n+1)^2$$

for $\delta \geqslant 2$. In particular, for $\delta = 3$ we have

$$\tfrac{3}{16} n^2 \leqslant B_{n,3} \leqslant \tfrac{3}{16}(n+1)^2.$$

Consequently, the estimation (2) is asymptotically best possible for $\delta = 3$. In a quite similar way, we obtain that the estimation (3) is asymptotically best possible for $\delta = 2$. Finally, let us observe that in the above defined graph G for $k = r$ we have $k = r = \gamma = \delta$, $\Delta = \delta^2$, and (v) yields $\gamma \leqslant g_{n,\delta}(\delta^2)/\delta^2 = \delta + 1/2$. This implies that the inequality (v) in Lemma 1 is, in some sense, best possible.

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